

Counting Problems

Counting problems are generally encountered somewhere in any mathematics course. Such problems are usually easy to state and even to get started, but how far they can be taken will vary widely, depending upon ability, interest and guidance. Just a few of the problems given here are finite and require little more than application and organisation to produce a correct count. Most though, start with an explicit requirement for a count of some particular cases and then lead on to ask for a 'count' that must be done by observing a pattern and extending a sequence. This can be done purely numerically in some cases but, more often it requires moving on to an algebraic approach.

Being organised and methodical is of considerable importance in this work. Devising and using some sort of notation is often of benefit too. Most of the problems given in this unit should be seen only as 'starting points'. They could be presented differently - since almost any one situation can lead to a lot of work, an explanation of what is required given from the front could lead to a whole lesson of activity. For any particular problem a lot more questions can be asked than the few given here. Only 'Triangles Galore' (page 9) has been covered in any depth as an example of what might be done. 'MacMahon Tiles' is an example of how these investigations can lead into an entirely different pastime.

Attention needs to be focused on the word 'different'. Exactly what it means varies with the context, sometimes it is obvious and sometimes it needs to be thought about. It may refer to shape, size, orientation, pattern, total or something else. Two shapes may be considered as different provided that there is at least one edge or face that they do not have in common. And then there is the matter of whether shapes or patterns are the same if one can be mapped into the other by means of a rotation or reflection. Again, in the absence of any specific instructions, context is all. Eventually a decision has to be made and perhaps sometimes a definition has to be created to be used throughout the counting.

Activities based on physical situations make it easy to see how a sequence is created and how it grows and, in a way, why differencing works. However, it is important to have examples to hand of cases where differencing does not work. The most obvious one is the list of prime numbers. One which arises from a physical situation is the growth of Polyominoes which produces the sequence starting

1, 1, 2, 5, 12, 35, 108, 369, 1285, 4655,

giving the number of polyominoes which can be made using

1, 2, 3, 4, 5, 6, 7, 8, 9 10 squares respectively.

There is no known formula or polynomial which will yield this sequence.

On the subject of sequences there is an excellent book which contains an ordered list of 5488 of the 'better known' sequences. It is

The Encyclopedia of Integer Sequences

by N J A Sloane & Simon Plouffe

Academic Press, 1995

ISBN 0-12-558630-2

But the 'master work' is available only on the Internet. It is on a Web-site run by N J A Sloane. The address is

<http://www.research.att.com/~njas/sequences/eisonline.html>

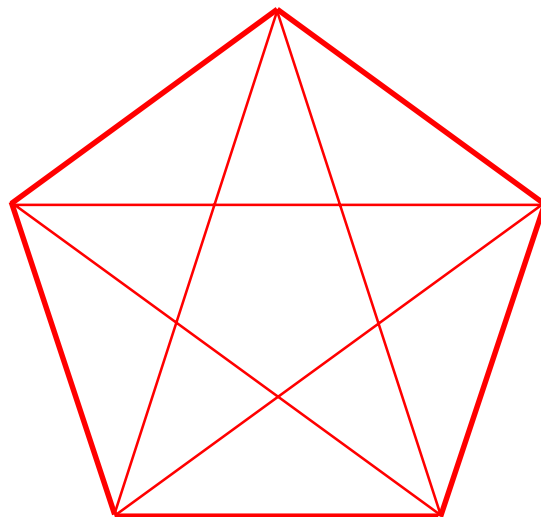
It contains over 50,000 sequences and is still growing!

The above address is a live link and (provided you are online) it can be called from here.

Be warned. It is possible to become addicted to this topic - inventing sequences (which is very easy) and then seeing if they are already known - and you can 'waste' a lot of time!

Triangulation

The drawing shows a regular pentagon (a polygon with 5 edges and vertices) with all its diagonals drawn in. The shape formed by the diagonals by themselves is known as a star polygon or (in this case) a pentagram.



How many triangles is it possible to trace out in this diagram?

The triangles do not have to be different in size.

Digit Count

If all of the numbers from 1 to 100 were written down, how many times would **each** of the digits (0, 1, 2, 8, 9) appear?

Partitions

Consider the number 4.
using the rules that

- only whole numbers (*not* including 0) may be used
- the only operation allowed is +
- order does not matter

(so $1 + 2 + 1$ is the same as $1 + 1 + 2$)

it could be written as

- (1) $1 + 1 + 2$
- (2) $2 + 2$
- (3) $1 + 3$
- (4) $1 + 1 + 1 + 1$
- (5) 4

and, as there are no other ways, it is then said that

“The number 4 can be **partitioned** in only five ways”.

How many ways can each of these numbers be partitioned

(a) 5 (b) 6 (c) 7 ?

Counting a Sequence

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6

The above sequence is made by writing: one 1, two 2's, three 3's, four 4's and so on.
The 10th number in the sequence is a 4, and the 20th number will be a 6.

Without writing the sequence in full, what will be

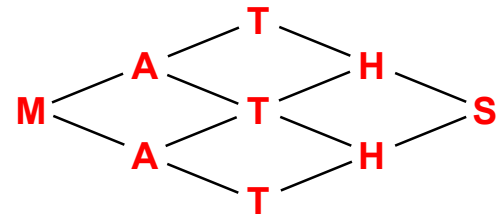
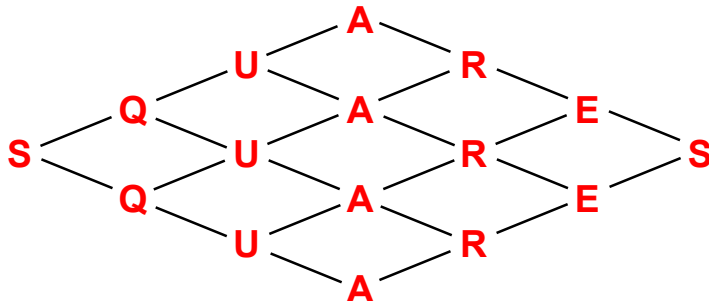
(a) the 50th (b) the 100th (c) the 1000th number?

Maths Spell

On these diagrams, the start position is the letter on the left, and the finish is the letter on the right. Moves must be from letter to letter, moving along the lines only.

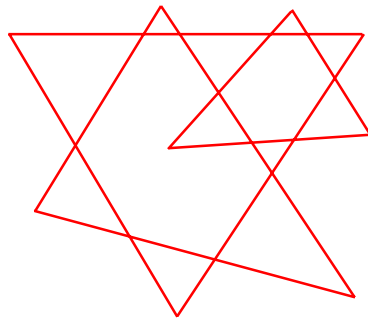
Following those rules, how many different routes can you find which spell out the words

- (a) **MATHS** (b) **SQUARES?**



Following the same idea, can you work out how many different routes would be possible in spelling the words

- (c) **EQUILATERAL**
(d) **CIRCUMFERENCE?**



Tricky Triangles

The diagram on the left is made up of three overlapping triangles.

(a) By moving only along the lines of the diagram, how many triangles can be traced out in total?

(b) Find an arrangement of three overlapping triangles which produces the greatest possible answer for (a).

Diagonals

A diagonal of a polygon is a straight line joining any two vertices which are not 'next-to' each other.

So,

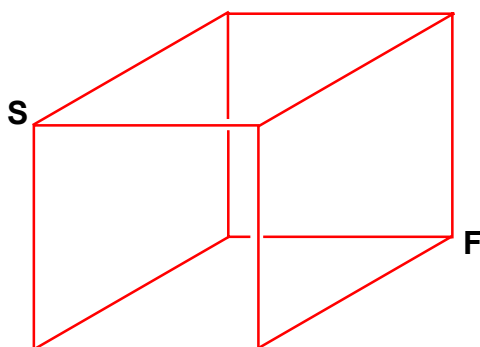
A triangle (3 vertices) has 0 diagonals.

A quadrilateral (4 vertices) has 2 diagonals.

A pentagon (5 vertices) has 5 diagonals.

How many diagonals will there be in a polygon having

- (a) 6 vertices (b) 7 vertices (c) 8 vertices?



Walking on Edge

The diagram is of a wire framework which forms the edges of a cube. The upper face (with vertex S) and the lower face (with vertex F) are both horizontal.

An insect is starting at S to walk to the finish at F.

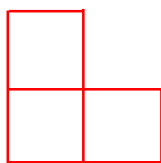
It may only walk along the edges, but must NOT walk along the same edge twice. It can only go DOWN the four vertical edges. Any vertex may be visited more than once, except for F.

In how many different ways can the walk be done?

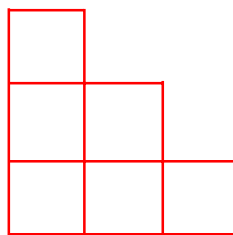
Stairs of Squares



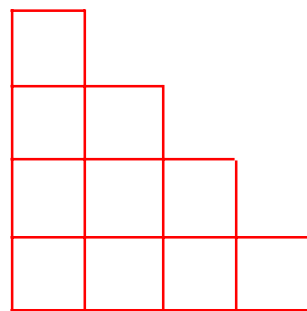
1



2



3



4

Some squares have been arranged to make 'staircases' as shown in the diagrams above.

Staircase No 1 needs 1 square

Staircase No 2 needs 3 squares

Staircase No 3 needs 6 squares . . . and so on

How many squares will be needed for the

(a) 5th

(b) 10th

(c) 20th

(d) 100th staircase?

Breaking-Up

The number 23 is to be 'broken-up' into five separate numbers which are joined by addition only. Only whole numbers are to be used (not including 0) and the five numbers must be different. The numbers must be written in size order so that a re-arrangement does not produce a different way.

Here are two ways it can be done

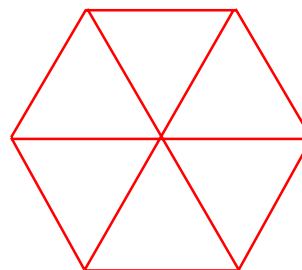
$$(1) 1 + 2 + 3 + 4 + 13 = 23$$

$$(2) 1 + 3 + 4 + 5 + 10 = 23$$

In how many different ways can it be done altogether?

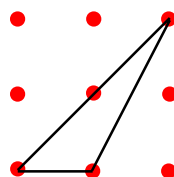
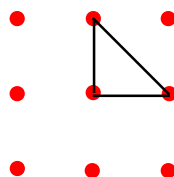
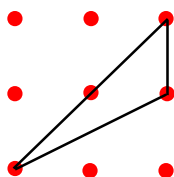
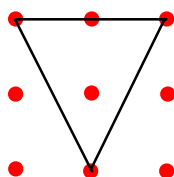
Colouring a Hexagon

A regular hexagon is divided up into six equilateral triangles. Each of these triangles is to be coloured either black or white. How many different ways can the colouring be done? A colouring is different if the finished hexagon cannot be turned around to match any other. Hexagons may not be turned over.



Dotty Triangles

Nine dots are set out in a square array with an equal distance between adjacent dots. Any three dots (which are not in a straight line) may be joined up to make a triangle. Four examples are shown.



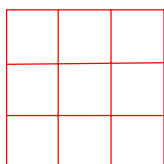
However, in this case we shall not consider these as being four different triangles since the second and the fourth are the same triangle turned around and turned over. So there are only three different triangles here.

How many different triangles is it possible to draw altogether?

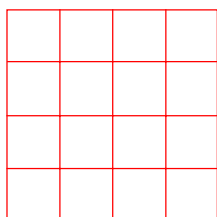
Square Counting

On each of these diagrams, using only the lines given,

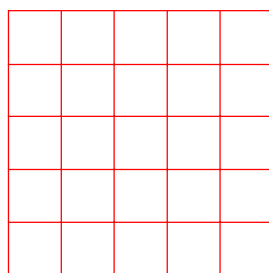
- How many differently-sized squares can be found
- How many different squares (of any size) can be traced out?



3 by 3



4 by 4



5 by 5

Example

In this diagram there are



- 2 differently-sized squares
- 5 different squares in total.

Without drawing it, can you say what the answers to (a) and (b) would be for a 10 by 10 grid?

Consecutives

Here are some sets of consecutive numbers and their totals

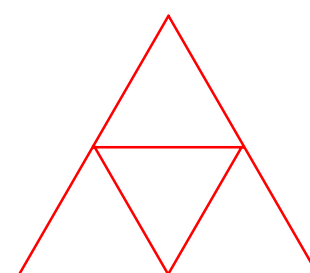
1st set	1	= 1
2nd set	2 + 3	= 5
3rd set	4 + 5 + 6	= 15
4th set	7 + 8 + 9 + 10	= 34
5th set	11 + 12 + 13 + 14 + 15	= 65

- How many numbers will there be in the 30th set?
- What number will the 10th set start with?
- What number will the 20th set end with?
- What will be the total of the
 - 10th
 - 20th
 - 50th set?

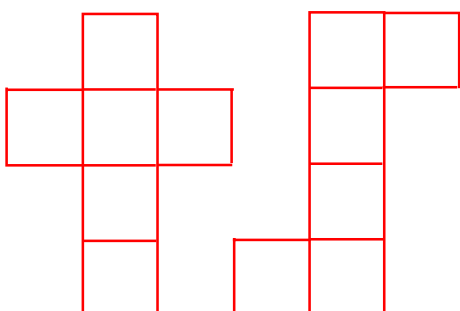
Colouration

The diagram is of a triangular tile, divided up into 4 separate equilateral triangles. The tile is to be coloured-in so that each of the 4 separate triangles is a different colour. Obviously, 4 different colours are needed.

- If only 4 different colours are available, in how many different ways may the tile coloured?
- If 5 different colours are available, in how many different ways may the tile be coloured then?
- ... and 6 colours?



In this case, a finished tile is different from any other if, by turning it around (but NOT over) it does not match any the pattern on any other tile.



Net Count

On the left are two different nets, each made of 6 squares joined at their edges. Each of these nets can folded up to make a cube. How many other different nets to make a square are possible, using 6 squares joined at their edges? A net is considered to be different if it cannot be turned around or turned over to fit any other net.

Rhyming and Reckoning

A Pupil's Lament

At subjects mathematical
I cannot get fanatical
It's all so problematical.

At homework arithmetical
I feel quite idiotical
And cannot get my sums right.
It's hard to see the difference
Between the geometrical
And what is trigonometrical.

As well as things statistical
There's algebra and all that stuff,
To me it's all identical.

I haven't understood a thing
So I cannot do my homework
And I'm bound to get an 'E' grade!

The 'poem' on the left was written to show one peculiarity. Each verse has three lines and each verse shows a different rhyming arrangement.

In verse 1 - all the lines rhyme

In verse 2 - only the first two lines rhyme

In verse 3 - only the last two lines rhyme

In verse 4 - only the first and last lines rhyme

In verse 5 - none of the lines rhyme.

One way of showing rhyming arrangements is by letters, where use of a repeated letter means that a rhyme exists. So the rhyming arrangements for these verses would be shown (in order) as

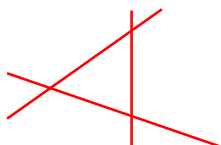
aaa aab abb abb aba abc

How many different rhyming arrangements are possible when there are **four** lines in each verse?

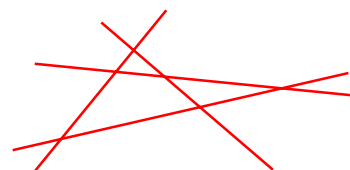
Maximum Triangles

For any given number of lines, how can they be arranged so that the greatest number of triangles can be traced out?

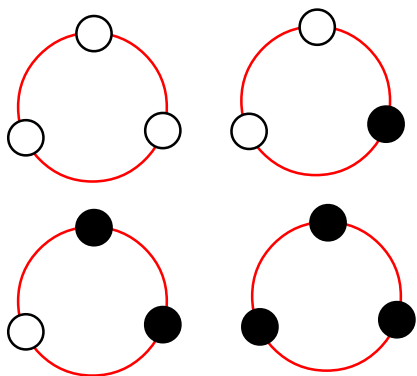
With 3 lines, only
1 triangle is possible



With 4 lines, it is
possible to find 4 triangles



If 5 lines are used, how many triangles are possible? And then 6 lines?



Necklaces

A necklace is to be made by putting some beads on a circle of wire. Only 2 colours of beads are available - black and white.

If the necklace is made up with 3 beads, then it can be done in only 4 different ways, and these are shown.

Note that, since the necklace is circular and the beads are free to move, it can be turned around or turned over, without making a different arrangement.

How many arrangements are possible for

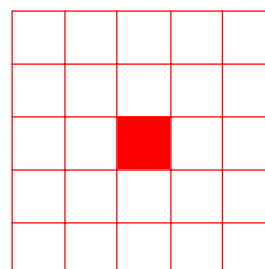
(a) 4 (b) 5 (c) 6 beads?

Quartering

A 5 by 5 square with a hole in the middle (as shown) is to be cut into four separate pieces - by cutting along the lines only.

The four pieces must be identical to each other in size and shape.

How many different ways are there of doing this?



Rings on Your Fingers

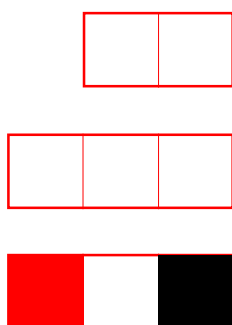
Imagine that all four fingers on your hand are the same size so that, given a ring to wear it could fit on any finger.

Now, given one ring, how many choices do you have for wearing it? Clearly, since it could be put on any finger, there are 4 choices.

What about two identical rings? If both rings are put on the same finger then there are still only 4 choices but, there is also the chance to wear one ring on one finger and the other ring on another finger. So how many choices are there altogether?

How many choices are there with three rings?

Tromino Colouring



The familiar domino is a rectangle divided into 2 squares.

The less-familiar tromino is a rectangle divided into 3 squares.

We wish to make a set of trominoes but, instead putting dots on them as is usual with dominoes, we intend that each square will be coloured. If we use 3 different colours it is possible to make a set of 18 different trominoes. On any single tromino a colour may appear any number of times

How many trominoes will there be in a set using 4 colours?

All Wrong!

Two (different) letters are written and have to go in the post, so two addressed envelopes are prepared. Now, in putting the letters into the envelopes there are 2 ways it can be done. In one way, each letter goes into the correct envelope, in the other way, each letter goes into the wrong envelope.

Suppose there were three letters to go into three envelopes. There are now 6 ways this can be done. Only 1 of those will have every letter in its correct envelope. In 3 of the ways, only one letter will be in its correct envelope. And in 2 of the ways every letter will be in the wrong envelope and we can say that they are 'ALL WRONG!'

With four letters and envelopes, there are 24 ways it can be done. How many of them are 'ALL WRONG!'

Noughts and Crosses

A completed grid in a game of Noughts and Crosses will have 5 O's and 4 X's - assuming O went first - which we will in all of this. Also, it is usual to stop as soon as a winning line has been made but, here we assume all 9 cells of the grid were filled. Four completed diagrams are shown below but there are only two different diagrams. The 3rd is only the 1st turned around and the 4th is only a reflection of the 1st.

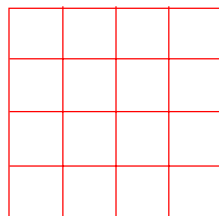
How many **different** final diagrams are possible?

X	O	X
O	O	X
O	X	O

O	X	O
X	O	X
O	X	O

O	O	X
X	O	O
O	X	X

X	O	X
X	O	O
O	X	O



Halving

A 4 by 4 square (as shown) is to be cut into two separate pieces - by cutting along the lines only. The two pieces must be identical to each other in size and shape.

How many different ways are there of doing this?

Cube Colouring

A cube is to have its six faces painted, each being either black or white.

If one of the faces is painted black then there is only one way of doing it.

But if two faces are black then, since those faces could be either opposite or next to each other, there are two different ways of doing it.

How many different ways are there altogether of colouring the cube?

Join the Dots

Around the circumference of a circle dots are marked, all an equal distance apart. Shapes are to be made by joining these dots by a set of straight lines so that every dot has two (and only two) lines connected to it.

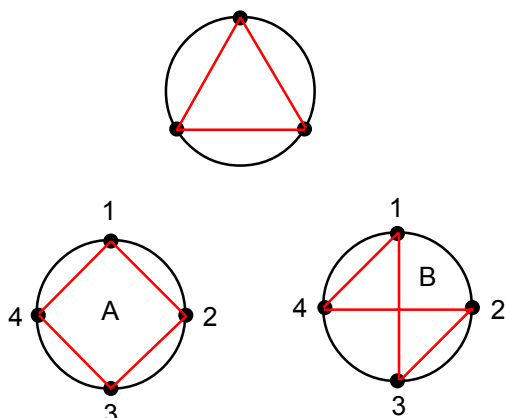
With three dots there is only one possibility.

With four dots there are two possibilities. Using the numbers shown, shape A could be described as (1 2 3 4 1) and the shape B as (1 3 2 4 1). It might be thought that another possibility is (1 2 4 3 1) but that is only shape B rotated.

With 5 dots there are 4 possibilities.

(a) Find and draw all 4

(b) Find, and draw, all the possibilities with 6 dots.



MacMahon Tiles

Sets of square tiles, each divided up into triangles by its two diagonals, are to be made by colouring in the triangles. All of the triangles must be coloured-in (no blanks) but they do not have to have different colours.

If there is only one colour available then there is only one possible tile that can be made - with all four triangles the same colour.

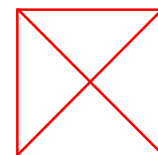
If there are two colours available then it is possible to make six different tiles.

(a) Find and draw the six possibilities.

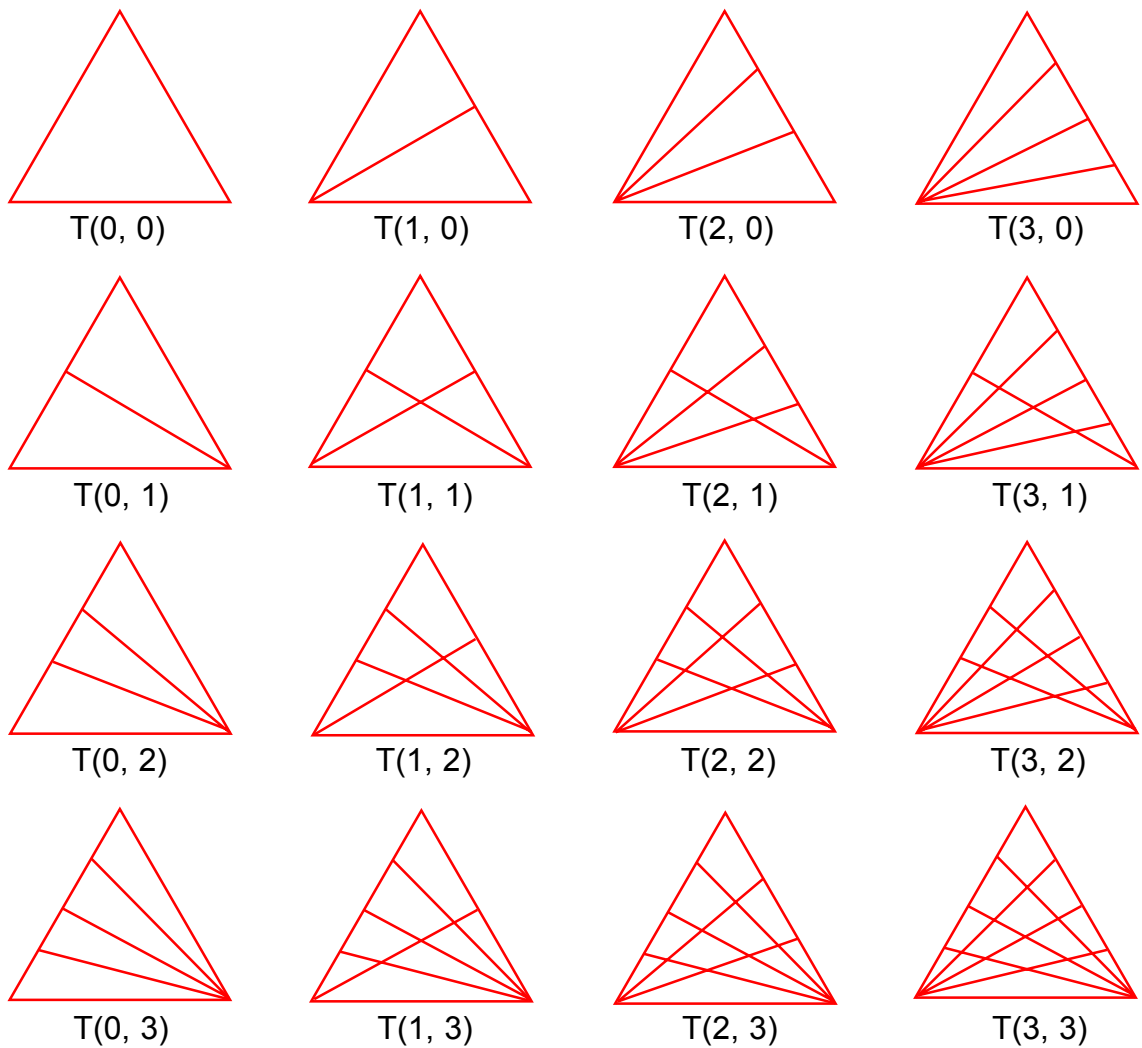
(b) How many different tiles are possible with three colours?

The set made with three colours are known as 'MacMahon Tiles' (after their discoverer). They have been produced and marketed commercially as a puzzle. The puzzle is to arrange them in a rectangle with the condition that tiles in contact with each other along an edge must match in colour along that edge. It is quite difficult, and often hard to believe that there are hundreds of solutions.

(c) Make a set of MacMahon Tiles and solve the puzzle.



Triangles Galore



Each of the triangles above is identified by $T(a, b)$.

a is a count of the number lines drawn from the bottom left vertex.

b is a count of the number of lines drawn from the bottom right vertex.

- (a) Draw $T(5, 0)$, $T(3, 5)$ and $T(4, 3)$.
- (b) How many triangles can be traced out in each of $T(1, 0)$, $T(0, 3)$, $T(2, 1)$, $T(4, 0)$ and $T(3, 2)$?
- (c) In which diagram can 8 triangles be found?
- (d) How many triangles could be found in $T(m, n)$?
- (e) What values of a and b would produce a diagram with 60 triangles?

Extensions

Consider the cases when lines are also drawn from the top vertex.

Do a similar investigation, searching for quadrilaterals.

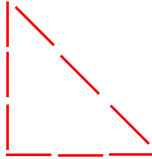
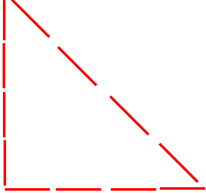
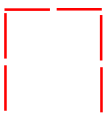
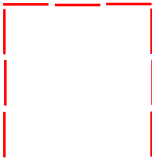
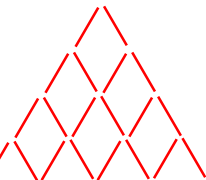
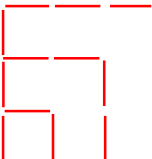
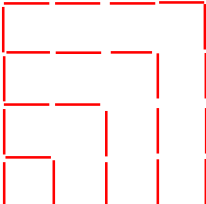
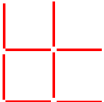
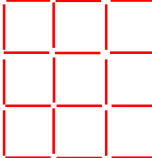
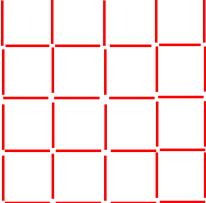
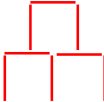
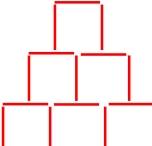
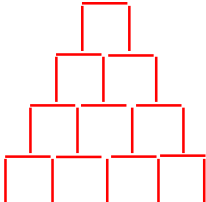
Matchstick Patterns

All of these diagrams could be made using matchsticks.

They are in sets, identified by a letter on the left and, in each set the first four of a sequence that is clearly growing are given. For each set, draw the 5th diagram.

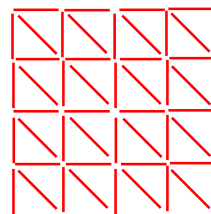
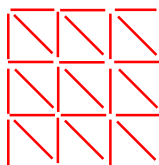
Say how many matchsticks would be needed to make

- (a) the 6th and 7th diagrams
- (c) the 10th diagram
- (d) the 20th, 50th and 100th diagrams.

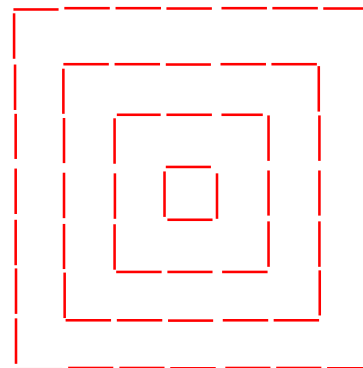
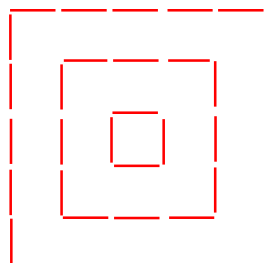
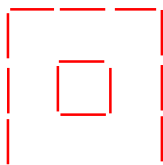
A				
B				
C				
D				
E				
F				
G				

Matchstick Patterns ~ 2

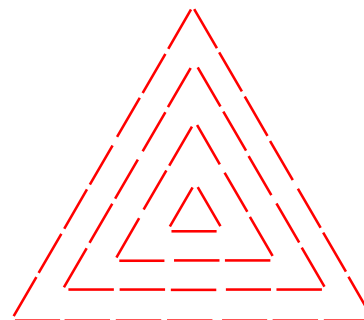
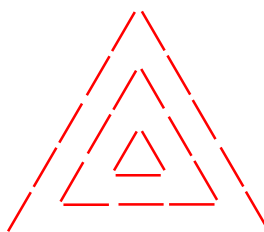
H



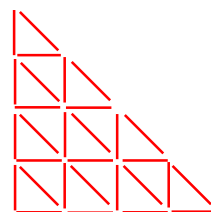
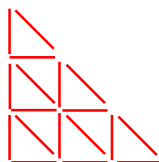
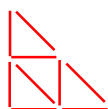
I



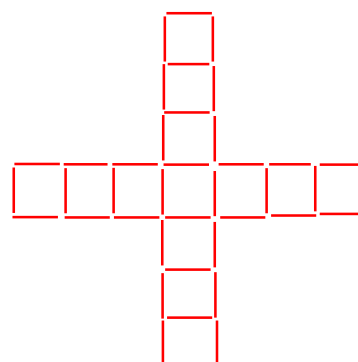
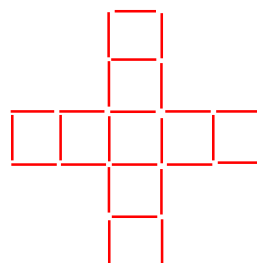
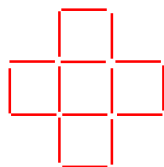
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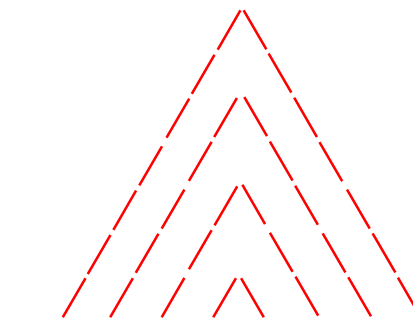
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L

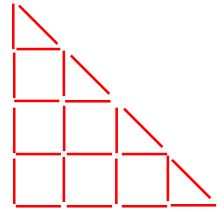
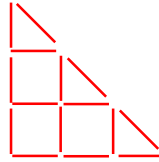
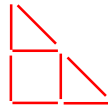


M

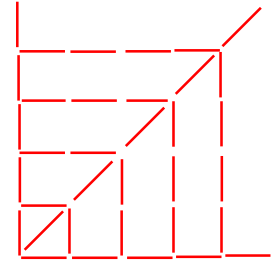
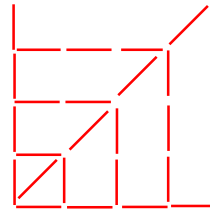
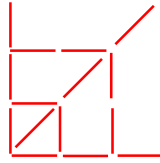
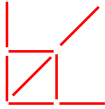


Matchstick Patterns ~ 3

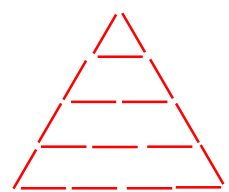
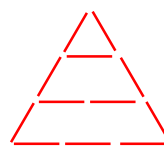
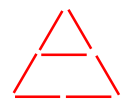
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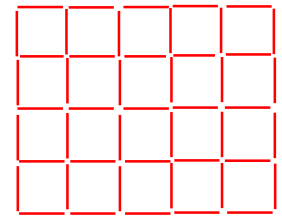
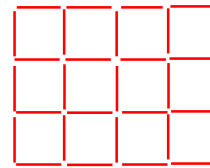
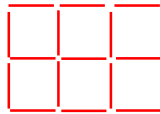
O



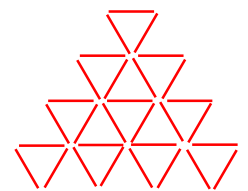
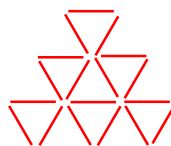
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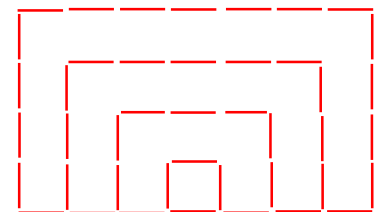
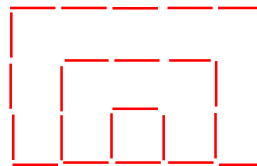
Q



D



S



T

